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Cosmic Microwave Background (Cont'd):

Setting up the equation for perturbations, we can now see how they evolve in the linear regime.

First, let us see what happens in the radiation-dominate phase.

We focus on perturbations in dark matter since due to zero pressure (after $t_{dec} \lesssim 1sec$), they have the greatest chance to grow.

In the radiation-dominated phase $\frac{\rho_{DM}}{\rho_0} \ll 1$, and $H(t) = \frac{1}{2t}$. As a result, we find the following equation for $\delta_{DM} \equiv \frac{\delta \rho_{DM}}{\rho_{DM}}$:

$$\ddot{\delta}_{DM} + \frac{1}{t}\dot{\delta}_{DM} = 0$$

We note that this is true for all modes. The solution to this equation is found to be:

$$\delta_{DM}(t) = \delta_{DM}(t_i) + \dot{\delta}_{DM}(t_i) \ln\left(\frac{t}{t_i}\right)$$

Here $\delta_{DM}(t_i)$ and $\dot{\delta}_{DM}(t_i)$ denote the amplitude and velocity of perturbations at an initial time t_i respectively. It is seen that even in the absence of the pressure term (as is the case for dark matter) perturbations can grow only logarithmically if they have a non-zero initial velocity. The growth in the radiation-dominate phase is therefore not significant.

The situation is different once the universe enters the matter-dominated phase (which happens at $t > t_{eq} \sim 50{,}000\ yr$). In this epoch $H(t) = \frac{2}{3t}$. Neglecting the contribution from baryons for simplicity (a more careful treatment to follow shortly), we find:

$$\ddot{\delta}_{DM} + \frac{4}{3t}\dot{\delta}_{DM} - \frac{2}{3t^2}\delta_{DM} = 0$$

This equation has a growing solution δ_+ and a decaying solution δ_-:

$$\delta_+(t) \sim \delta_+(t_i)\left(\frac{t}{t_i}\right)^{\frac{2}{3}}, \qquad \delta_-(t) \sim \delta_-(t_i)\left(\frac{t}{t_i}\right)^{-1}$$

The growing solution is expected because of the attractiveness of of gravity that tends to increase the density in overdense regions. The decaying solution, however, may look curious. It corresponds to the case when perturbations have an initial velocity $\dot{\delta}(t_i)$ that is negative in overdense regions (i.e., where $\delta(t_i) > 0$). The general solution is a linear combination of the growing and decaying solutions:

$$\delta(t) = a\,\delta_+(t_i) + b\,\delta_-(t_i) \simeq a\,\delta_+(t_i) \qquad t \gg t_i$$

We note that in a static background $\delta_+(t)$ is an exponential function of time. In an expanding universe, however, the damping caused by the expansion results in a slower growth that is power law in nature.

Now that we know perturbations grow in power law fashion in the matter-dominated phase, let us consider a multi-component fluid that describes the early universe. We consider a fluid with thre components: dark matter, baryons, and radiation. We would like to see how dark matter and baryon perturbations evolve in the interval $t_{eq} \lesssim t \lesssim t_{rec}$. Within this interval, dark matter is pressureless, while radiation (i.e., photons) provide a pressure to the baryonic component with a speed of sound $v_s = \sqrt{\frac{1}{3}}$. The equations governing dark matter perturbations $\delta_{DM} \equiv \frac{\delta \rho_{DM}}{\rho_{M}}$ and baryon perturbations $\delta_B \equiv \frac{\delta \rho_B}{\rho_B}$ are:

$$\begin{cases} \ddot{\delta}_{DM} + \dfrac{4}{3t}\dot{\delta}_{DM} - \dfrac{2}{3t^2}\left(\varepsilon_{DM}\delta_{DM} + \varepsilon_B \delta_B\right) = 0 \\ \ddot{\delta}_B + \dfrac{4}{3t}\dot{\delta}_B + \dfrac{k^2}{3t^{\frac{4}{3}}}\delta_B - \dfrac{2}{3t^2}\left(\varepsilon_{DM}\delta_{DM} + \varepsilon_B\delta_B\right) = 0 \end{cases} \qquad (I)$$

Here $\rho_0 \simeq \rho_{DM} + \rho_B$ and $\varepsilon_{DM} \sim \frac{5}{6}$, $\varepsilon_B \sim \frac{1}{6}$ (note that $\rho_{DM} \sim 5\rho_B$).

We assume adiabatic initial conditions $\delta_{DM}(t_i) = \delta_B(t_i) \sim 10^{-5}$. This is the simplest predictions of inflationary models that is in agreement with current observations.

As discussed before, baryon perturbations start in the unstable regime where pressure can be neglected. Therefore, for better clarity, we consider the two regimes $\lambda \gg \lambda_J$ (unstable) and $\lambda < \lambda_J$ (stable) separately.

Unstable Regime:

For $\lambda \gg \lambda_J$ ($k < k_J$) the pressure term in the second equation in (I) is negligible. We then find:

$$\begin{cases} \ddot{\delta}_{DM} + \dfrac{4}{3t}\dot{\delta}_{DM} - \dfrac{2}{3t^2}\left(\varepsilon_{DM}\delta_{DM} + \varepsilon_B\delta_B\right) = 0 \\ \ddot{\delta}_B + \dfrac{4}{3t}\dot{\delta}_B - \dfrac{2}{3t^2}\left(\varepsilon_{DM}\delta_{DM} + \varepsilon_B\delta_B\right) = 0 \end{cases} \qquad (\mathrm{II})$$

It is seen from (II) that for adiabatic initial conditions δ_{DM} and δ_B evolve the same in the unstable regime. We can then define:

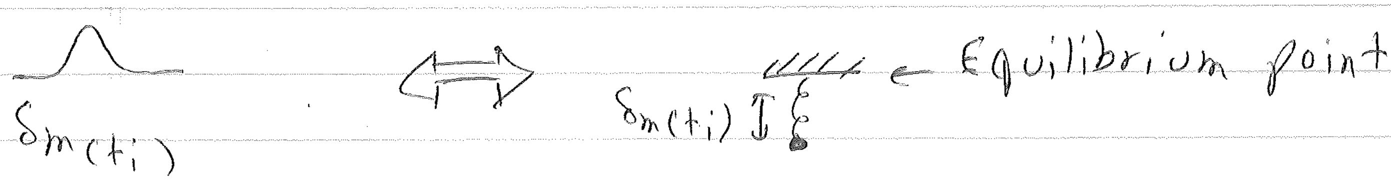
$$\delta_m \equiv \frac{\delta\rho_{DM} + \delta\rho_B}{\rho_{DM} + \rho_B} = \delta_{DM} = \delta_B$$

By adding the two equations in (II) we obtain the following equation for δ_m:

$$\ddot{\delta}_m + \frac{4}{3t}\dot{\delta}_m - \frac{2}{3t^2}\delta_m = 0$$

This results in $\delta_m(t) \propto \left(\frac{t}{t_i}\right)^{\frac{2}{3}}$ (as seen before).

One can make a useful analogy with the harmonic oscillator since this is just the equation for a damped harmonic oscillator. Initial value of perturbations $\delta_m(t_i)$ correspond to the initial displacement of a harmonic oscillator from its equilibrium point.



Perturbations grow in the unstable regime, which corresponds to an inverted harmonic oscillator, for which the displacement from the equilibrium point increases: